CLOSURE, INTERIOR, BOUNDARY

DEFINITIONS: Suppose $(X.\mathcal{T})$ is a topological space and $A \subseteq X$.

- The **closure** of A, denoted $\overline{A}$ or $\mathrm{cl}_X(A)$ is defined as follows:

$$\overline{A} = \bigcap \{C \ : \ A \subseteq C \text{ and } C \text{ is closed in } X\}$$

- The **interior** of A, denoted A° or $\mathrm{int}_X(A)$ is defined as follows:

$$\overline{A} = \bigcup \{T \ : \ T \subseteq A \text{ and } T \text{ is open in } X\}$$

- The **boundary** of A, denoted ∂A or $\mathrm{bd}_X A$ is: $\partial A = \overline{A} \cap \overline{X \backslash A}$

- The **exterior** of A, denoted ext_X is $\mathrm{ext}_X = (X \backslash A)^\circ$

EXAMPLE: Prove the following properties:

- $\overline{A}$ is closed. Moreover, $\overline{A}$ is the 'smallest' closed superset of A; i.e., if $A \subseteq C$ and C is closed, then $\overline{A} \subseteq C$.
- A° is open. Moreover, A° is the 'largest' open subset of A; i.e., if $T \subseteq A$ and T is open, then $T \subseteq A^\circ$.

EXAMPLE: Let $(\mathbb{R}, \mathcal{E})$ be the real numbers with the Euclidean topology.

Calculate the closure, interior, and boundary of the following subsets of $\mathcal{R}$.

- $\emptyset$
- $(0, 1)$
- $[0, 1)$
- $[0, 1]$
- $\mathbb{N}$
- $\left\{\frac{1}{n} \ : \ n \in \mathbb{N}\right\}$
- $\mathbb{Q}$
- $\mathbb{R} \backslash \mathbb{Q}$
- $\mathbb{R}$

DEFINITION: A subset $A \subseteq X$ in a space $(X, \mathcal{T})$ is **dense** if $\overline{A} = X$.

EXAMPLE: Suppose $(X, \mathcal{T})$ is a topological space and $A \subseteq X$. Prove the following:

- A is closed iff $A = \overline{A}$.
- $\overline{A} = \{x \,:\, \text{if } x \in T \text{ and } T \text{ is open, then } T \cap A \neq \emptyset\}$

  NOTE: It may be easier to work with the complements.
- A is open iff $A = A^\circ$.
- $A^\circ = \{x \,:\, \exists T \text{ open such that } x \in T \subseteq A\}$
- ∂A is closed.
- $\partial A = \overline{A} \backslash A^\circ$.
- $\partial A = \{x \,:\, \text{if } x \in T \text{ and } T \text{ is open, then } T \cap A \neq \emptyset \text{ and } T \cap (X \backslash A) \neq \emptyset\ \}$
- $X = \text{int}_X(A) \,\dot{\cup}\, \text{bd}_X(A) \,\dot{\cup}\, \text{ext}_X(A)$.
- $X \backslash \overline{A} = (X \backslash A)^\circ$, hence, $\overline{A} = X \backslash\, (X \backslash A)^\circ$
- $X \backslash A^\circ = \overline{X \backslash A}$, hence $A^\circ = X \backslash \overline{(X \backslash A)}$

PROJECT IDEA: Research 'Kuratowski's Closure - Complement Problem.'

EXPLORATION: What, if any, is the relationship between the following sets:

- $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$?
- $\overline{A \cup B}$ and $\overline{A} \cup \overline{B}$?
- $(A \cap B)^\circ$ and $A^\circ \cap B^\circ$
- $(A \cup B)^\circ$ and $A^\circ \cup B^\circ$

EXPLORATION: Suppose $A \subseteq X$ and $B \subseteq A$:

- What, if any, is the relationship between $\text{cl}_A(B)$ and $\text{cl}_X(B)$?
- What, if any, is the relationship between $\text{int}_A(B)$ and $\text{int}_X(B)$?

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces. Prove:

- $F : X \to Y$ is continuous iff $F(\overline{A}) \subseteq \overline{F(A)}$.
- $F : X \to Y$ is continuous iff $F^{-1}(A^\circ) \subseteq \left[F^{-1}(A)\right]^\circ$.

EXPLORATION: Can you formulate and prove results similar to the one above for open functions? Closed?